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ANALYTIC SOLUTIONS TO A CLASS OF TWO-DIMENSIONAL LOTKA-VOLTERRA DYNAMICAL SYSTEMS

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Abstract. A novel coordinate transformation is used to reduce a simple generalization of the Lotka-Volterra dynamical system to a single second-order autonomous ordinary differential equation. Formal analytic solutions to this differential equation are presented which are shown to reduce to the recently obtained solution to the Lotka-Volterra system [C. M. Evans and G. L. Findley, J. Math. Chem. 25 (1999) 105–110]. An initial analysis of the analytic solution to this latter system results in the specification of a new family of Lotka-Volterra related differential equations.

1. Introduction. The Lotka-Volterra (LV) system consists of the following pair of first-order autonomous ordinary differential equations:

$$\dot{x}_1 = a x_1 - b x_1 x_2 , \qquad \dot{x}_2 = -c x_2 + b x_1 x_2 , \tag{1}$$

where $x_1(t)$ and $x_2(t)$ are real functions of time, $\dot{x}_i = dx_i/dt$, and a, b, c are positive real constants. This system was originally introduced by Lotka [7] in 1920 as a model of undamped oscillations in autocatalytic chemical reactions, and was later applied by Volterra [14] to treat predator-prey interactions in ecology. Other applications have followed in the intervening years in physics [11], chemistry [9], population biology [13] and epidemiology [12].

Since the original publication by Lotka [7], it has been known that equations (1) possess the dynamical invariant

$$\Lambda = b(x_1 + x_2) - \ln x_1^c x_2^a .$$

By means of a logarithmic transformation, Kerner [4] showed that Λ serves to reduce equations (1) to a Hamiltonian system. This has recently sparked a resurgence of interest in the LV problem (including a rediscovery of some previously known results [5,10]), particularly with regard to dynamical invariants of generalizations of equations (1) [6,11].

In a recent study [2], we used Λ to define a novel coordinate transformation (see Section 2, below) that leads to analytic solutions of equations (1) in the form of integral quadratures. We were also able to use this transformation to define

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a new family of Lotka-Volterra related differential equations (see Section 5, below), all of which admitted analytic solutions in the form of integral quadratures [3]. It is of interest, then, to explore the applicability of this coordinate transformation to a wider class of LV generalizations.

One immediate generalization of the LV system is [8]

$$\dot{x}_1 = a x_1 - b x_1 x_2^m , \qquad \dot{x}_2 = - c x_2 + b x_1^n x_2 \tag{2}$$

where a, b, c are again positive real constants, and m, n are nonzero real constants. In analogy to the LV invariant, equations (2) can easily be shown to possess the dynamical invariant

$$\Lambda = \frac{b}{m\, n} (m x_1^n + n x_2^m) - \ln(x_1^c x_2^a) . \tag{3}$$

Obviously, equations (2) reduce to the original LV system for the case $m = n = 1$. In the present paper, we apply our earlier coordinate transformation [2] to equations (2) in Section 2, and thereby generate analytic solutions to these equations in Section 3. In Section 4, we show the equivalence of the results obtained here to our earlier results [2] for the LV problem ($m = n = 1$), and describe an initial formal analysis of the analytic solution to this latter problem. Finally, in Section 5 we summarize the development of a new family of LV related differential equations [3] that is based upon the results of Section 4.

2. Coordinate transformation. Our solution to equations (2) starts by choosing new coordinates z_1 and z_2 such that $z_1^2 + z_2^2 = 1$. One choice for these new coordinates is

$$z_1 = \frac{1}{\sqrt{\Lambda}} \left[\frac{b}{m\, n} (m x_1^n + n x_2^m) \right]^{1/2} , \quad z_2 = \frac{1}{\sqrt{\Lambda}} \left[- a \ln (x_1^\alpha x_2) \right]^{1/2}$$

where $\alpha = c/a$, and the positive square root is implied in each case. In terms of z_1 and z_2, x_1 and x_2 are

$$x_1 = \left[\xi\, n\, (a \alpha\, m z_1^2 - 2 z_2 \dot{z}_2) \right]^{1/n} , \quad x_2 = \left[\xi\, m\, (a\, n z_1^2 + 2 z_2 \dot{z}_2) \right]^{1/m} , \tag{4}$$

where $\xi = \Lambda / [b\, a\, (n + \alpha\, m)]$.

Since $z_1^2 + z_2^2 = 1$, an angle ϕ can be defined such that $z_1 = \sin \phi$ and $z_2 = \cos \phi$. Using equations (2) and (4), then, we find the characteristic differential equation

$$\begin{aligned} \ddot{\phi} &+ [\cot \phi - \tan \phi - 2 b\, \xi\, m\, n \cos \phi \sin \phi]\, \dot{\phi}^2 \\ &+ a (n - \alpha\, m) [b\, \xi\, m\, n \sin^2 \phi - 1]\, \dot{\phi} \\ &+ (1/2)\, a^2 \alpha\, m\, n \tan \phi\, [b\, \xi\, m\, n \sin^2 \phi - 1] = 0 , \end{aligned}$$

which is equivalent to the dynamical system of equations (2). By making the substitution $w = b\, \xi\, m\, n\, (1 - \cos 2 \phi)/2$, the characteristic differential equation simplifies to

$$\ddot{w} - \dot{w}^2 + a (\alpha\, m - n) (1 - w)\, \dot{w} - a^2 \alpha\, m\, n\, w\, (1 - w) = 0 , \tag{5}$$

and the solutions to equations (2) are given by

$$x_1 = \left[\frac{1}{b\,m}(a\,\alpha\,m\,w + \dot{w})\right]^{1/n} , \quad x_2 = \left[\frac{1}{b\,n}(a\,n\,w - \dot{w})\right]^{1/m} . \tag{6}$$

In the next Section, equation (5) is solved by integral quadrature.

3. Analytic solutions. Obtaining a solution to equation (5) begins by rewriting equation (3) as

$$b^m x_2^{\,m} = (-1)^m k^{2m} b^{-m} x_1^{-\alpha m} \mathrm{e}^{b(m x_1^{\,n} + n x_2^{\,m})/a n} , \quad k^2 = -b^2 \mathrm{e}^{-\Lambda/a} ,$$

or, after substituting equations (6) and rearranging,

$$a\,n\,w - \dot{w} = (-1)^m \frac{b^{1-2m}}{n} k^{2m}\left[\frac{1}{b m}(a\,\alpha\,m\,w + \dot{w})\right]^{-m\alpha/n} \mathrm{e}^{[(\alpha m + n) w/n]} . \tag{7}$$

By defining a new function $\rho(w)$ such that

$$\alpha\, a m w + \dot{w} = a\,\alpha\,m\,\mathrm{e}^{\rho} , \tag{8}$$

and writing $a\,n\,w - \dot{w} = a\,(\alpha\,m + n)\,w - (a\,\alpha\,m\,w + \dot{w})$, equation (7) simplifies to

$$a\,(\alpha\,m + n)\,w - a\,\alpha\,m\,\mathrm{e}^{\rho} + (-1)^{m+1} k^{2m} n\, b^{1-2m}\left(\frac{a\,\alpha}{b}\right)^{-m\alpha/n} \mathrm{e}^{-m\alpha\rho/n}\, \mathrm{e}^{[(\alpha m + n) w]/n} = 0 .$$

Since $\rho(w)$ can be determined in principle from the last equation, and since the first integral of equation (5) is given by equation (8), the integral quadrature resulting from equation (8), namely,

$$t - t_0 = \int_w \left[\alpha\, a\, m\,(\mathrm{e}^{\rho} - w')\right]^{-1} d\,w' , \tag{9}$$

provides the analytic solution to equation (5) and, thereby, the analytic solution to equations (2). Finally, by substituting equation (8) into equations (6), x_1 and x_2 become

$$x_1 = \left(\frac{a\,\alpha}{b} e^{\rho}\right)^{1/n} , \quad x_2 = \left(\frac{a}{b\,n}\left[(n - \alpha\,m)\,w - \alpha\,m\,e^{\rho}\right]\right)^{1/m} . \tag{10}$$

In the remainder of this paper, we focus on the special case $m = n = 1$, which represents the original LV system of equations (1).

4. Lotka-Volterra system. Our previously presented [2] analytic solution to equations (1) is

$$t - t_0 = \int_w \left[\alpha\, a\,(\mathrm{e}^{\rho} - w')\right]^{-1} d\,w' , \tag{11}$$

where $\rho(w)$ solves

$$b\,a\,(\alpha + 1)\,w - b\,\alpha\,a\,\mathrm{e}^{\rho} + k^2\left(\frac{\alpha\,a}{b}\right)^{-\alpha} \mathrm{e}^{(\alpha+1)w}\, \mathrm{e}^{-\alpha\rho} = 0 ,$$

and x_1 and x_2 are given by

$$x_1 = \frac{a\,\alpha}{b}\,e^{\rho} \ , \quad x_2 = \frac{a}{b}\,[(1-\alpha)\,w - \alpha\,e^{\rho}] \ .$$

Clearly, this solution is identical to the results of Section 3 for $m = n = 1$.

An initial formal analysis of equation (11) begins by rewriting equation (3) ($m = n = 1$) as

$$b\,x_1\,x_2 = -\frac{k^2}{b}\,x_1^{1-\alpha}\,\mathrm{e}^{b(x_1+x_2)/a} \ . \tag{12}$$

Using equation (12), equations (1) can be rearranged to give

$$\begin{aligned} \dot{x}_1 &= a\,x_1 + \frac{k^2}{b}\,x_1^{1-\alpha}\,\mathrm{e}^{b(x_1+x_2)/a} \\ \dot{x}_2 &= -a\,\alpha\,x_2 - \frac{k^2}{b}\,x_1^{1-\alpha}\,\mathrm{e}^{b(x_1+x_2)/a} \ , \end{aligned} \tag{13}$$

which can then be substituted into equations (6) to yield, after some manipulation [3],

$$\ddot{w} + a(\alpha-1)\,\dot{w} - a^2\alpha\,w - k^2\left[\frac{1}{b}(a\,\alpha\,w + \dot{w})\right]^{(1-\alpha)}\mathrm{e}^{(\alpha+1)w} = 0 \ .$$

A power series expansion of the exponential term in the last equation gives

$$\ddot{w} + a(\alpha-1)\,\dot{w} - a^2\alpha\,w - k^2\left[\frac{1}{b}(a\,\alpha\,w + \dot{w})\right]^{(1-\alpha)}\sum_{j=0}^{\infty}\frac{1}{j!}(\alpha+1)^j\,w^j = 0 \ ,$$

which can be truncated to yield

$$\ddot{w} + a(\alpha-1)\,\dot{w} - a^2\alpha\,w - k^2\left[\frac{1}{b}(a\,\alpha\,w + \dot{w})\right]^{(1-\alpha)}\sum_{j=0}^{l}\frac{1}{j!}(\alpha+1)^j\,w^j = 0 \ , \tag{14}$$

where $l = 0, \ldots, \infty$. The solutions to equation (14) represent, for each finite l, an approximate solution to equations (13), with $l = \infty$ corresponding to the exact solution. The analytic solutions to equation (14) have the form of equation (11), with $\rho(w)$ now being determined by [3]

$$(l+1)! \times \left[b\,a(\alpha+1)\,w - b\,\alpha\,a\,\mathrm{e}^{\rho} + k^2\left(\frac{\alpha\,a}{b}\right)^{-\alpha}\mathrm{e}^{-\alpha\rho}\sum_{j=0}^{l+1}\frac{1}{j!}(\alpha+1)^j\,w^j\right] = 0 \ .$$

In some instances, equation (14) can be solved in terms of known functions. For example, when $\alpha = 1$, solving the last equation for $\rho(w)$ and substituting this result into equation (11) gives the integral quadrature [3]

$$t - t_0 = \pm\int_w\left[a^2\,w'^2 + k^2\sum_{j=0}^{l+1}\frac{2^j}{j!}\,w'^j\right]^{-1/2} d\,w' \ .$$

With the use of a symbolic processor, this equation can be integrated in terms of known functions for $l \le 3$ [3]. The general form of these solutions is exponential when $l \leq 1$ (although the solution for $l = 1$ can become periodic when $a^2 < 2\,k^2$) and is an elliptic function of the first kind when $1 < l \leq 3$ [3]. (Other values of α are also explored in [3].)

The solutions to equation (14) represent analytic solutions to a family of differential equations which are related to the LV system [3]. As shown in the next Section, this family can be derived from an inverse transformation of equations (6), coupled with the knowledge of equation (14).

5. Systems of LV related differential equations. Inverting equations (6) yields

$$w = \frac{b}{a}(\alpha + 1)^{-1}(x_1 + x_2)\ , \quad \dot{w} = b\,(\alpha + 1)^{-1}(x_1 - \alpha\, x_2)\ .$$

Substituting $\ddot{w}$ obtained from equation (14) into the time derivative of equations (6), and employing the above inverse transformation yields the following system of first-order autonomous ordinary differential equations [3]:

$$\begin{aligned} \dot{x}_1 &= a\,x_1 + \frac{k^2}{b}x_1^{1-\alpha}\sum_{j=0}^{l}\frac{1}{j!}\left(\frac{b}{a}\right)^j(x_1 + x_2)^j\ , \\ \dot{x}_2 &= -\,a\,\alpha\,x_2 - \frac{k^2}{b}x_1^{1-\alpha}\sum_{j=0}^{l}\frac{1}{j!}\left(\frac{b}{a}\right)^j(x_1 + x_2)^j\ . \end{aligned} \tag{15}$$

Although these equations appear to be more complicated than the original LV system (and in fact are equivalent to that system when $l = \infty$), they can be solved in terms of known functions when $\alpha = 1$ and $l \leq 3$, as shown above in Section 4. Equations (15) can possibly be used as new models for biological or chemical systems, since these equations contain both the quadratic coupling term which appears in the LV system, as well as quadratic terms dependent only upon x_1 and x_2 (which is reminiscent of the LV competition model [1]).

The dynamical invariant [3] for equations (15) can be obtained by integrating

$$\frac{dx_1}{dx_2} = \frac{a\,x_1 + \dfrac{k^2}{b}x_1^{1-\alpha}\displaystyle\sum_{j=0}^{l}\frac{1}{j!}\left(\frac{b}{a}\right)^j(x_1 + x_2)^j}{-\,a\,\alpha\,x_2 - \dfrac{k^2}{b}x_1^{1-\alpha}\displaystyle\sum_{j=0}^{l}\frac{1}{j!}\left(\frac{b}{a}\right)^j(x_1 + x_2)^j}$$

to give

$$\Lambda_l = a\,x_1^{\alpha}\,x_2 + k^2\sum_{j=1}^{l+1}\frac{1}{j!}\,a^{1-j}\,b^{j-2}(x_1 + x_2)^j\ .$$

We have also found a simple transformation which converts this invariant into Hamiltonian form [3].

6. Conclusions. The LV dynamical system, and generalizations of this system, continue to generate interest 80 years after the original introduction of this model by Lotka [7]. In this paper, we presented analytic solutions to a generalization of the LV system, as well as to the LV system itself, which resulted from a novel coordinate transformation involving the dynamical invariant for the system. We then described an initial formal analysis of the analytic solution to the LV system [2,3]. Finally, the development of a new family of differential equations [3] that results from our analysis of the LV system was reviewed.

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